# LINEARIZABILITY CRITERIA FOR SYSTEMS OF TWO SECOND-ORDER DIFFERENTIAL EQUATIONS BY COMPLEX METHODS

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Abstract. Lie's linearizability criteria for scalar second-order ordinary differential equations had been extended to systems of second-order ordinary differential equations by using geometric methods. These methods not only yield the linearizing transformations but also the solutions of the nonlinear equations. Here complex methods for a scalar ordinary differential equation are used for linearizing systems of two second-order ordinary and partial differential equations, which can use the power of the geometric method for writing the solutions. Illustrative examples of mechanical systems including the Lane-Emden type equations which have roots in the study of stellar structures are presented and discussed.

## 1. Introduction

Lie developed a systematic procedure for solving nonlinear ordinary differential equations (ODEs) with some minimal symmetry under transformations of the dependent and independent variables, called *point transformations*, by using group theory [22]. For second-order ODEs he provided criteria for their being transformable to linear ODEs, provided they are maximally symmetric [23]. The requirement is that they be at most cubically nonlinear in the first derivative and that their coefficients satisfy a set of four constraints for the first derivatives, involving two auxiliary functions. Tressé [37, 38] reduced the number of constraining equations to two for higher derivatives by eliminating the auxiliary functions.

Since Lie's time there have been various developments following up his work. There was the development of contact transformations (see e.g., [7, 18]) and their use for linearizing two classes of third-order ODEs [16, 17]; approach of Cartan type used for the same purpose [10, 11]; a method for a special case of a class of third-order scalar ODEs [32]; Lie's algebraic methods used for classification of systems of ODEs [25, 26, 39]; extension to partial differential equations (PDEs) using potential symmetries and otherwise [5, 6, 21]; the use of algebraic computing to extend Lie's methods to general third- and fourth-order ODEs [19, 20, 33]; conditional linearizability [28, 29]; geometric methods for symmetry analysis [4, 13, 35] and their use for linearization [30, 31]; and the development of complex symmetry analysis [1, 2, 3]. Here we use the last two developments to provide linearization for a class of systems of two PDEs and an alternative method for linearizing a class of systems of two ODEs that is not equivalent to the earlier method for the same purpose [3].

The geometric approach for linearization of a system of two ODEs requires that the system be at most cubic in its first derivatives and satisfy a generalized set of Lie-Tressé invariant conditions which are written in terms of the coefficients of the system of equations [30, 31]. This requirement comes from regarding the system as a projection of the geodesic equations in a flat space (in curvilinear coordinates). The linearizing transformation is then obtained

*Key words and phrases.* complex linearization, complex Lie symmetries, Lie linearizability .

by converting the metric in the given coordinates to one in Cartesian coordinates. This enables us to write down the solution of the nonlinear equations directly. In complex symmetry analysis one studies the relationship between the algebraic properties of the complex differential system and the corresponding real system of ordinary or partial differential equations, including the Cauchy-Riemann equations, which arises from the complex system via complex splitting of the dependent and independent variables [1, 2, 3]. We present linearization criteria, both invariant and algebraic, for systems of two real ODEs and PDEs that arise from the linearization of scalar complex second-order ODEs. This also provides an answer to the inverse problem of those systems of ODEs and PDEs that arise from the linearization of scalar complex second-order ODEs. Furthermore, we provide examples of mechanical systems that are linearizable by our procedure.

The plan of the paper is as follows. In the next Section the salient points of complex symmetry analysis and the geometric methods developed are mentioned. In Section 3 invariant linearizability criteria for systems of two PDEs are given and some physical examples provided. In the subsequent section restricted complex transformations is introduced and used to yield invariant linearizability criteria for systems of two ODEs. Again, illustrative examples of mechanical systems are provided. Section 5 consists of a discussion and conclusion.

## 2. Preliminaries

A scalar ODE for a complex function of a complex variable can be written in terms of the real and imaginary parts of the dependent variable as functions of the real and imaginary parts of the independent variable. As such it yields a pair of PDEs. Of course we need to include the Cauchy-Riemann equations (C-REs), which guarantee the differentiability of the dependent complex variable in our system. As such we get a system of four PDEs, two of which are first-order equations. The symmetries of the original ODE are not identical with those of the system of PDEs. Nevertheless the solutions of the ODE obtained by using its symmetries give the solution of the PDE. As stated this may seem trivial. However, one could pick a system of PDEs and check if it corresponds to an ODE. In the present note we also specify those systems of PDEs that could be transformed to such complex differential equations. Then one can use the solution of the ODE to write a solution of the system of PDEs. The correspondence between the symmetries of the system of PDEs and the ODE is the subject of "complex symmetry analysis" [1, 2]. Thus, for example, the single infinitesimal generator for a function of one variable, $\mathbf{Z} = \partial/\partial z$ becomes the pair of generators $\mathbf{X} = \partial/\partial x$ and $\mathbf{Y} = \partial/\partial y$, where $z$ is the complex independent variable, $x$ is its real part and $y$ its imaginary part. Again the scaling symmetry for the independent complex variable gives the scaling symmetry for the two real variables together along with the rotation generator in the 2-dimensional space. Similar remarks apply for the generators involving the dependent complex variable or a mixture of the two.

For a linear scalar ODE the conversion to the complex form and thence to the system of two PDEs is trivial. The only complication is that here the system becomes a set of four equations, the two C-REs and the two PDEs for the two real variables corresponding to the single complex variable. This can change the symmetry structure but does not make any other substantive difference. For a nonlinear ODE the situation changes. Now there can be non trivial mixing between the real and imaginary parts of the complex dependent

variable so that the system of two PDEs becomes significantly coupled. The question can arise whether the C-REs continue to hold under some transformation of the independent and dependent variables. This is of relevance for us as we need to use linearizing transformations. Suppose the ODE is written for a complex analytic function of a single variable, $u(z)$. Since the linearizing point transformation $\mathcal{L} : (z, u) \to (Z, U)$ is analytic (by definition), the real linearizing transformation $\mathcal{RL} : (x, y, f, g) \to (X, Y, F, G)$ satisfies the C-REs. Note that when we go to deal with systems of ODEs, the transformed variables are *not* guaranteed to satisfy the C-REs rather they satisfy a partial analytic structure. In that case we have to use the C-REs in the original variables. This creates complications in the linearization of those ODEs. Apart from this complexity the solution of nonlinear system is still attained through a procedure analogous to analytic continuation.

The system of geodesic equations is of second-order and quadratically semi linear in the first derivatives and has no other terms in it. It inherits the isometries, but can have many other symmetries. It was noted [4] that projection of this system, using the translational invariance symmetry of the geodesic parameter yields a system of cubically semi linear ODEs. The linearizability of a quadratically semi linear system of ODEs of geodesic type is provided by regarding the coefficients of the quadratic terms as Christoffel symbols and verifying whether the resulting Riemann tensor is zero or not [30]. If the system is linearizable, one can find the linearizing transformation by taking the coordinate transformation from the metric tensor constructed from the Christoffel symbols [14] to the metric tensor in Cartesian coordinates. The same procedure can be extended to the projected system of cubically semi linear ODEs [31]. The power of this method is apparent from the fact that one can not only find the linearizing transformations but also be able to write down the solution of the nonlinear equation.

## 3. Invariant Linearizability Criteria for Systems of PDEs

First-order scalar ODEs are always linearizable. We only need to deal with second-order systems for checking linearizability. The geometric linearization of systems of two ODEs [30, 31] gives a requirement that the system be (at most) cubically semi linear in the dependent variables and satisfy a generalization of the Lie conditions coming from the flatness of the space in which the geodesics lie. However, this procedure does not cover all the classes of linearizable systems as it requires a $15-$dimensional symmetry algebra, whereas there can also be $5-, 6-, 7-$ and $8-$dimensional algebras of linearizable systems [39]. Since the Lie symmetry algebra for linear PDEs is infinite dimensional in general, the question arises whether we can expect the PDEs to be (at most) cubically semi linear as well. The geometric approach relied on the connection between geometry and systems of second-order ODEs via the system of geodesic equations [4, 13]. As there is no apparent geometrical way of extending the Lie conditions to systems of PDEs, it is not clear what the analogues of the linearizability criteria would be.

We extend Lie's linearizability criteria to a class of systems of two PDEs obtainable from complex scalar ODEs. The real transformations for linearization of this system of nonlinear PDEs can be obtained by decomposing the complex transformations that linearize the complex ODE. We consider a simple equation to illustrate this. For example, we exploit the real

transformation,

$$F=\frac{f}{f^2+g^2}-x,\ G=\frac{-g}{f^2+g^2}-y, \tag{1}$$

to map a nonlinear coupled first-order system of PDEs,

$$\begin{aligned} f_x+g_y&=-2f^2+2g^2,\\ g_x-f_y&=-4fg, \end{aligned} \tag{2}$$

into a linear system

$$\begin{aligned} F_x+G_y&=0,\\ G_x-F_y&=0. \end{aligned} \tag{3}$$

Note that the real transformation (1) is equivalent to a complex transformation

$$U=(1/u)-z. \tag{4}$$

Therefore we can employ complex variables to map systems of nonlinear PDEs into their simpler forms. It is important to mention that the linearization above is different from the classical way of linearizing equations because the real transformation (1) is obtained from the complex Lie transformation (4) that may or may not be a real Lie transformation. Therefore complex linearizability works in a different way. A natural question following the above discussion would be "which systems of PDEs can be treated via complex linearization?". We provide a class of systems of second-order PDEs which upon satisfying a set of four conditions is subject to complex linearizability in Theorem 1. It further provides a partial answer to another important question, i.e. "which systems of PDEs correspond to complex differential equations?" It is indispensable to use complex linearization in several cases because it plays a crucial role in extracting solutions of systems of differential equations that would have been difficult otherwise. We highlight this feature of complex variables in the examples. In the next Section we extend the above characteristic of complex transformations in dealing with systems of ODEs. Namely it is shown that, if the above transformation is restricted on a single line, then a nonlinear system of first-order ODEs can also be linearized. In general restricted complex transformations can be brought into play in the linearization of systems of ODEs.

For the above purpose we first prove a few results that are used to obtain linearizability of this class of systems of PDEs.

**Theorem 1**. *The class of systems of two second-order partial differential equations*

$$\begin{aligned} f_{xx}-f_{yy}+2g_{xy}&=4w_1(x,y,f,g,h,l),\\ g_{xx}-g_{yy}-2f_{xy}&=4w_2(x,y,f,g,h,l), \end{aligned} \tag{5}$$

*where*

$$2h=f_x+g_y,\ 2l=g_x-f_y, \tag{6}$$

*is complex-linearizable if and only if the functions,* $w_1$ *and* $w_2$*, are at most cubic in* $h$ *and* $l$*, i.e.*

$$\begin{aligned}
f_{xx}-f_{yy}+2g_{xy} &= 4A^1h^3-12A^1hl^2-12A^2h^2l+4A^2l^3+4B^1h^2-4B^1l^2-\\
&\qquad 8B^2hl+4C^1h-4C^2l+4D^1,\\
g_{xx}-g_{yy}-2f_{xy} &= 12A^1h^2l-4A^1l^3+4A^2h^3-12A^2hl^2+8B^1hl+4B^2h^2-\\
&\qquad 4B^2l^2+4C^2h+4C^1l+4D^2
\end{aligned}\tag{7}$$

*together with the constraints on the coefficients*

$$\begin{aligned}
&3A^1_{xx}-3A^1_{yy}+6A^2_{xy}+6C^1A^1_x+6C^1A^2_y-6A^2_xC^2+6C^2A^1_y-\\
&6A^1_fD^1-6D^1A^2_g+6D^2A^2_f-6D^2A^1_g+6A^1C^1_x+6A^1C^2_y-6A^2C^2_y+\\
&6A^2C^1_x+C^1_{ff}-C^1_{gg}+2C^2_{fg}-12A^1D^1_f-12A^1D^2_g+12A^2D^2_f-12A^2D^1_g+\\
&2B^1C^1_f+2B^1C^2_g-2B^2C^2_f+2B^2C^1_g-4B^1B^1_x-4B^1B^2_y+4B^2B^2_x-4B^2B^1_y-\\
&2B^1_{xf}-2B^2_{yf}-2B^2_{xg}+2B^1_{yg}=0,\\
&3A^2_{xx}-3A^2_{yy}-6A^1_{xy}+6C^2A^1_x+6C^2A^2_y+6A^2_xC^1-6C^1A^1_y-\\
&6D^2A^1_f-6D^2A^2_g-6D^1A^2_f+6D^1A^1_g+6A^2C^1_x+6A^2C^2_y+6A^1C^2_y-\\
&6A^1C^1_x+C^2_{ff}-C^2_{gg}-2C^1_{fg}-12A^2D^1_f-12A^2D^2_g-12A^1D^2_f+12A^1D^1_g+\\
&2B^2C^1_f+2B^2C^2_g+2B^1C^2_f-2B^1C^1_g-4B^2B^1_x-4B^2B^2_y-4B^1B^2_x+\\
&4B^1B^1_y-2B^2_{xf}+2B^1_{yf}+2B^1_{xg}-2B^2_{yg}=0,\\
&12D^1A^1_x+12D^1A^2_y-12D^2A^2_x+12D^2A^1_y-6D^1B^1_f-6D^1B^2_g+\\
&6D^2B^2_f-6D^2B^1_g+6A^1D^1_x+6A^1D^2_y-6A^2D^2_x+6A^2D^1_y+\\
&B^1_{xx}-B^1_{yy}+2B^2_{xy}-2C^1_{xf}-2C^2_{yf}-2C^2_{xg}+2C^1_{yg}-6B^1D^1_f-\\
&6B^1D^2_g+6B^2D^2_f-6B^2D^1_g+3D^1_{ff}-3D^1_{gg}+6D^2_{fg}+4C^1C^1_f+\\
&4C^1C^2_g-4C^2C^2_f+4C^2C^1_g-2C^1B^1_x-2C^1B^2_y+2C^2B^2_x-2C^2B^1_y=0,\\
&12D^2A^1_x+12D^2A^2_y+12D^1A^2_x-12D^1A^1_y-6D^2B^1_f-6D^2B^2_g-\\
&6D^1B^2_f+6D^1B^1_g+6A^2D^1_x+6A^2D^2_y+6A^1D^2_x-6A^1D^1_y+\\
&B^2_{xx}-B^2_{yy}-2B^1_{xy}-2C^2_{xf}+2C^1_{yf}+2C^1_{xg}+2C^2_{yg}-6B^2D^1_f-\\
&6B^2D^2_g-6B^1D^2_f+6B^1D^1_g+3D^2_{ff}-3D^2_{gg}-6D^1_{fg}+4C^2C^1_f+\\
&4C^2C^2_g+4C^1C^2_f-4C^1C^1_g-2C^2B^1_x-2C^2B^2_y-2C^1B^2_x+2C^1B^1_y=0,
\end{aligned}\tag{8}$$

*where all the coefficients* $A^i, B^i, C^i$ *and* $D^i$, $(i=1,2)$, *are functions of* $x, y, f$ and $g$.

**Proof.** We firstly assume an analytic structure on the manifold and that there exists a complex transformation that project the functions $f(x,y)$ and $g(x,y)$ to a single complex

function $u$ of complex variable $z$. Furthermore assume that there exists four complex functions, $A, B, C$ and $D$, such that

$$\begin{aligned}
A(x,u) &= A^1(x,y,f,g) + iA^2(x,y,f,g),\\
B(x,u) &= B^1(x,y,f,g) + iB^2(x,y,f,g),\\
C(x,u) &= C^1(x,y,f,g) + iC^2(x,y,f,g),\\
D(x,u) &= D^1(x,y,f,g) + iD^2(x,y,f,g).
\end{aligned} \tag{9}$$

By invoking (9) we can map the system (7) to a second-order complex differential equation

$$u''(x) = A(x,u)u'^3 + B(x,u)u'^2 + C(x,u)u' + D(x,u), \tag{10}$$

which is at most cubic in its first derivative. Therefore it satisfies Lie's linearizability criteria. Moreover the conditions (8) can be projected down to a set of two equations

$$\begin{aligned}
&3A_{zz} + 3A_zC + 3AC_z - 3A_uD + C_{uu} - 6AD_u + BC_u - 2BB_z - 2B_{zu} = 0,\\
&6A_zD - 3B_uD + 3AD_z + B_{zz} - 2C_{zu} - 3BD_u + 3D_{uu} + 2CC_u - CB_z = 0;
\end{aligned} \tag{11}$$

which may be recognized as the Lie compatibility conditions. As the ODE (10) is linearizable, so the system of PDEs (7) is also linearizable.

**Theorem 2.** *If the system of PDEs* (5) *admits four real symmetries* $\mathbf{X}_1$, $\mathbf{Y}_1$, $\mathbf{X}_2$ *and* $\mathbf{Y}_2$, *such that*

$$\mathbf{X}_1 = \rho_1\mathbf{X}_2 - \rho_2\mathbf{Y}_2,\ \mathbf{Y}_1 = \rho_1\mathbf{Y}_2 + \rho_2\mathbf{X}_2, \tag{12}$$

*for nonconstant* $\rho_1$ *and* $\rho_2$, *and their commutators satisfy*

$$[\mathbf{X}_1,\mathbf{X}_2] - [\mathbf{Y}_1,\mathbf{Y}_2] = 0,\ [\mathbf{X}_1,\mathbf{Y}_2] + [\mathbf{Y}_1,\mathbf{X}_2] = 0, \tag{13}$$

*then, there exists a point transformation* $(x,y,f,g) \longrightarrow (X,Y,F,G)$, *which reduces* $\mathbf{X}_1$, $\mathbf{Y}_1$,$\mathbf{X}_2$ *and* $\mathbf{Y}_2$ *to their canonical form*

$$\mathbf{X}_1 = \frac{\partial}{\partial F},\quad \mathbf{Y}_1 = \frac{\partial}{\partial G},\quad \mathbf{X}_2 = X\frac{\partial}{\partial F} + Y\frac{\partial}{\partial G},\quad \mathbf{Y}_2 = Y\frac{\partial}{\partial F} - X\frac{\partial}{\partial G}, \tag{14}$$

*and the system* (5) *can be reduced to the linear form*

$$\begin{aligned}
F_{XX} - F_{YY} + 2G_{XY} &= 4W_1(X,Y),\\
G_{XX} - G_{YY} - 2F_{XY} &= 4W_2(X,Y).
\end{aligned} \tag{15}$$

**Proof.** Suppose that $\mathbf{X}_a + i\mathbf{Y}_a = \mathbf{Z}_a$, for $a = 1,2$. Then equation (13) can be replaced by $[\mathbf{Z}_1,\mathbf{Z}_2] = 0$, which implies that the two complex symmetries $\mathbf{Z}_1$, $\mathbf{Z}_2$, commute with each other. Further, setting $\mathbf{Z}_1 = \rho(z,u)\mathbf{Z}_2$ for a nonconstant complex function $\rho$, justifies equation (12). It is proved by Lie (see [24]) that every scalar second-order ODE (10) that admits two commuting symmetries such that $\mathbf{Z}_1 = \rho(z,u)\mathbf{Z}_2$ can be transformed into a linear ODE $U'' = W(\zeta)$, by applying the point transformation, $\zeta = \zeta(z,u), U = U(z,u)$ which reduces $\mathbf{Z}_1$ and $\mathbf{Z}_2$ to their canonical forms

$$\mathbf{Z}_1 = \frac{\partial}{\partial U},\ \mathbf{Z}_2 = \zeta\frac{\partial}{\partial U}. \tag{16}$$

The point transformation, $(x,y,f,g) \longrightarrow (X,Y,F,G)$, can be obtained for complex transformation which then can be used to convert system (5) into the linear form (15).

**Examples**

**1.** *Higher-dimensional Coupled System of Modified Lane-Emden Type:*

The Lane-Emden equation arises in the study of stellar structures [9]. We first investigate the linearizability of a system of two cubically semi linear PDEs

$$\begin{aligned} f_{xx} - f_{yy} + 2g_{xy} &= -12fh + 12gl - 4f^3 + 12fg^2, \quad f_x = g_y, \\ g_{xx} - g_{yy} - 2f_{xy} &= -12gh - 12fl - 12f^2g + 4g^3, \quad f_y = -g_x. \end{aligned} \tag{17}$$

The above system can be regarded as a special case of a higher-dimensional Lane-Emden system equipped with an analytic structure. The coefficients satisfy linearizability conditions (8). Therefore the above system is linearizable. Thus one can check the linearizability of a class of systems of PDEs analogously to Lie's technique. In order to construct the transformation we employ the symmetries of (17). It admits Lie symmetries

$$\begin{aligned} \mathbf{X}_1 &= \frac{\partial}{\partial x}, \quad \mathbf{Y}_1 = \frac{\partial}{\partial y}, \\ \mathbf{X}_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - f\frac{\partial}{\partial f} - g\frac{\partial}{\partial g}, &\quad \mathbf{Y}_2 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} - g\frac{\partial}{\partial f} + f\frac{\partial}{\partial g}, \end{aligned} \tag{18}$$

which satisfy all the conditions of Theorem 2. Therefore (18) leads to the linearizing transformation,

$$\begin{aligned} X &= x - \frac{f}{f^2+g^2}, \quad Y = y + \frac{g}{f^2+g^2}, \\ F &= \frac{1}{2}(x^2-y^2) - \frac{f^2+g^2}{xf+yg}, \quad G = xy - \frac{f^2+g^2}{yf-xg}, \end{aligned} \tag{19}$$

and we deduce the following system of linear PDEs

$$\begin{aligned} F_{XX} - F_{YY} + 2G_{XY} &= 0, \\ G_{XX} - G_{YY} - 2F_{XY} &= 0. \end{aligned} \tag{20}$$

**2.** Consider the nonlinear anharmonic oscillator system given by the PDEs

$$\begin{aligned} f(f_{xx} - f_{yy} + 2g_{xy}) - g(g_{xx} - g_{yy} - 2f_{xy}) &= 4(h^2 - l^2) - 4(f^2 - g^2)w_1 + 8fgw_2, \\ f(g_{xx} - g_{yy} - 2f_{xy}) + g(f_{xx} - f_{yy} + 2g_{xy}) &= 8hl - 8fgw_1 - 4(f^2 - g^2)w_2, \\ f_x = g_y, &\quad f_y = -g_x, \end{aligned} \tag{21}$$

where both $w_1$ and $w_2$ are arbitrary functions of $x$ and $y$. We can transform the above system into a system of the form (7). Consequently it can be verified that the coefficients of $h$ and $l$ satisfy the conditions (8). Therefore the above system is linearizable. Notice that we can convert the system (21) into the second-order complex nonlinear anharmonic oscillator ODE

$$uu'' + w(z)u^2 = u'^2. \tag{22}$$

It admits the symmetries

$$\mathbf{Z}_1 = zu\frac{\partial}{\partial u}, \quad \mathbf{Z}_2 = u\frac{\partial}{\partial u}, \tag{23}$$

which yields the complex transformation

$$Z = \frac{1}{z}, \quad U = \frac{1}{z}\log u. \tag{24}$$

The above transformation gives the real transformation

$$X = \frac{x}{x^2+y^2}, \quad Y = \frac{-y}{x^2+y^2},$$
$$F = \frac{1/2}{x^2+y^2}\left(x\ln\left(f^2+g^2\right)+2y\arctan\left(g/f\right)\right),$$
$$G = \frac{1/2}{x^2+y^2}\left(2x\arctan\left(g/f\right)-y\ln\left(f^2+g^2\right)\right), \tag{25}$$

that transforms system (21) into the linear form

$$F_{XX}-F_{YY}+2G_{XY} = -\frac{4}{X^2+Y^2}\left((X^3-3XY^2)w_1-(Y^3-3X^2Y)w_2\right),$$
$$G_{XX}-G_{YY}-2F_{XY} = -\frac{4}{X^2+Y^2}\left((X^3-3XY^2)w_2+(Y^3-3X^2Y)w_1\right), \tag{26}$$

where

$$w_1 = w_1\left(\frac{X}{X^2+Y^2}, \frac{-Y}{X^2+Y^2}\right),$$
$$w_2 = w_2\left(\frac{X}{X^2+Y^2}, \frac{-Y}{X^2+Y^2}\right). \tag{27}$$

**3.** We use the transformation

$$X = 2f-x^2+y^2, \; Y = 2g-2xy, \; F = x, \; G = y, \tag{28}$$

to linearize the system of PDEs

$$f_{xx}-f_{yy}+2g_{xy} = 4+4\left((h-x)^2-(l-y)^2\right)w_1-8(h-x)(l-y)w_2,$$
$$g_{xx}-g_{yy}-2f_{xy} = 8(h-x)(l-y)w_1+4\left((h-x)^2-(l-y)^2\right)w_2,$$
$$f_x = g_y, \quad f_y = -g_x, \tag{29}$$

where $w_1$ and $w_2$ are arbitrary functions of the mixed variables $2f-x^2+y^2$ and $2g-2xy$. The above system possesses the symmetries

$$\mathbf{X}_1 = \frac{\partial}{\partial x}+x\frac{\partial}{\partial f}+y\frac{\partial}{\partial g}, \quad \mathbf{Y}_1 = x\frac{\partial}{\partial g}-y\frac{\partial}{\partial f}-\frac{\partial}{\partial y},$$
$$\mathbf{X}_2 = x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+(x^2-y^2)\frac{\partial}{\partial f}+2xy\frac{\partial}{\partial g},$$
$$\mathbf{Y}_2 = y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}+2xy\frac{\partial}{\partial f}-(x^2-y^2)\frac{\partial}{\partial g}. \tag{30}$$

The linearized system of PDEs is

$$F_{XX}-F_{YY}+2G_{XY} = -2(Hw_1-Lw_2),$$
$$G_{XX}-G_{YY}-2F_{XY} = -2(Hw_2+Lw_1). \tag{31}$$

These conclude our examples. In the next Section we present another important use of the complex method.

## 4. Restricted Complex Transformations and Invariant Linearizability Criteria for Systems of Two ODEs

In the previous section we utilized complex functions of complex variables to obtain linearizing transformations for systems of PDEs. If we restrict our complex functions to depend upon a single real variable, then it would generate different transformations. They transform systems of ODEs into other systems of ODEs. We extend our three-dimensional space of two dependent and one independent variables to a two-complex-dimensional space. In the intermediate steps we move off the real line to obtain our linearizing transformations. Then the solutions of systems of ODEs are recovered by restricting the independent variable to the real line. The procedure is reminiscent of analytic continuation. In the process we loose the C-REs so our system is not the same as that with which we started. It is very interesting to see the invariance of analytic structure under the restricted complex transformation. In this case the ODE is written for an analytic function of a single real variable, $u(x)$. It is important to understand how the complex transformation works in the restricted domain. The linearizing complex point transformation $\mathcal{L}:(x,u)\rightarrow(\chi,U)$ can exhibit dual nature in terms of its analyticity which yields the following real linearizing transformation $\mathcal{RL}:(x,f,g)\rightarrow(\chi,\Upsilon,\zeta)$. Because in the transformed variables $\chi$ can be either complex or real. If it is real, then the transformed variables satisfy C-REs only in $f$ and $g$ in which case there is a partial analytic structure on the transformed variables, but, if $\chi$ is complex, then the complete analytic structure is restored on the transformed variables. We illustrate this important feature in the two examples below. Thus we cannot guarantee that at the end a similar linearized system can be obtained via the linearizing transformations from other approaches even though we do get the solution by linearization (in the complex).

The invariance properties of a system of ODEs,

$$\begin{aligned} f'' &= w_1(x,f,g,f',g'),\\ g'' &= w_2(x,f,g,f',g'), \end{aligned} \tag{32}$$

have been investigated by using complex functions in [1, 2]. The idea is to make use of the transformation,

$$u(x)=f(x)+ig(x),\ \ w(x,u)=w_1(x,f,g)+iw_2(x,f,g), \tag{33}$$

to convert the system (32) into the single ODE,

$$u''(x)=w(x,u,u'), \tag{34}$$

and then use the standard Lie procedure of linearization. We call this procedure *complex linearization* even though we have no guarantee that the transformed system can be linearized via other approaches. To comprehend the connection between complex transformations and linearization we firstly revisit an earlier example from the previous section.

Consider a first-order two-dimensional Riccati system

$$\begin{aligned} f' &= -f^2+g^2,\\ g' &= -2fg, \end{aligned} \tag{35}$$

which is a special case of a general Riccati system in two dimensions. A natural question arises: which transformation can linearize the above system? If such a transformation exists,

then how can we find it. The use of the transformation,

$$\Upsilon = \frac{f}{f^2+g^2} - x, \quad \zeta = \frac{-g}{f^2+g^2}, \tag{36}$$

maps system (35) into the simplest system

$$\Upsilon' = 0, \quad \zeta' = 0. \tag{37}$$

The transformation (36) *is* a mere consequence of the same complex transformation (4) that we used to linearize system of PDEs (2) in the remaining Section. The only difference is that we restricted the dependent variable to a single real line. This indicates to us a significant use of restricted complex transformations in converting systems of nonlinear equations into their linear analogues. Theorem 3 provides a class of systems of two second-order ODEs that can be dealt via complex variables. We now present the basic theorem for complex linearization.

**Theorem 3.** *The necessary and sufficient condition for a system of two second-order ODEs of the form*

$$\begin{aligned} f'' &= A^1f'^3 - 3A^1f'g'^2 - 3A^2f'^2g' + A^2g'^3 + B^1f'^2 - B^1g'^2 - 2B^2f'g' + \\ &\qquad C^1f' - C^2g' + D^1, \\ g'' &= 3A^1f'^2g' - A^1g'^3 + A^2f'^3 - 3A^2f'g'^2 + 2B^1f'g' + B^2f'^2 - B^2g'^2 + \\ &\qquad C^2f' + C^1g' + D^2, \end{aligned} \tag{38}$$

*where* $A^i$, $B^i$, $C^i$, $D^i$, $(i = 1, 2)$ *are functions of the variables* $x, y, f$ *and* $g$, *to be solvable by complex linearization is that the coefficients satisfy the conditions*

$$\begin{aligned} &12A^1_{xx} + 12C^1A^1_x - 12A^2_xC^2 - 6A^1_fD^1 - 6D^1A^2_g + 6D^2A^2_f - 6D^2A^1_g + \\ &\quad 12A^1C^1_x - 12A^2C^2_x + C^1_{ff} - C^1_{gg} + 2C^2_{fg} - 12A^1D^1_f - 12A^1D^2_g + \\ &12A^2D^2_f - 12A^2D^1_g + 2B^1C^1_f + 2B^1C^2_g - 2B^2C^2_f + 2B^2C^1_g - 8B^1B^1_x + \\ &\qquad 8B^2B^2_x - 4B^1_{xf} - 4B^2_{xg} = 0, \\ &12A^2_{xx} + 12C^2A^1_x + 12A^2_xC^1 - 6D^2A^1_f - 6D^2A^2_g - 6D^1A^2_f + 6D^1A^1_g + \\ &\quad 12A^2C^1_x + 12A^1C^2_x + C^2_{ff} - C^2_{gg} - 2C^1_{fg} - 12A^2D^1_f - 12A^2D^2_g - \\ &12A^1D^2_f + 12A^1D^1_g + 2B^2C^1_f + 2B^2C^2_g + 2B^1C^2_f - 2B^1C^1_g - 8B^2B^1_x - \\ &\qquad 8B^1B^2_x - 4B^2_{xf} + 4B^1_{xg} = 0, \\ &24D^1A^1_x - 24D^2A^2_x - 6D^1B^1_f - 6D^1B^2_g + 6D^2B^2_f - 6D^2B^1_g + \\ &\quad 12A^1D^1_x - 12A^2D^2_x + 4B^1_{xx} - 4C^1_{xf} - 4C^2_{xg} - 6B^1D^1_f - 6B^1D^2_g + \\ &\quad 6B^2D^2_g - 6B^2D^1_g + 3D^1_{ff} - 3D^1_{gg} + 6D^2_{fg} + 4C^1C^1_f + 4C^1C^2_g - \\ &\qquad 4C^2C^2_f + 4C^2C^1_g - 4C^1B^1_x + 4C^2B^2_x = 0, \\ &24D^2A^1_x + 24D^1A^2_x - 6D^2B^1_f - 6D^2B^2_g - 6D^1B^2_f + 6D^1B^1_g + \\ &\quad 12A^2D^1_x + 12A^1D^2_x + 4B^2_{xx} - 4C^2_{xf} + 4C^1_{xg} - 6B^2D^1_f - 6B^2D^2_g - \\ &\quad 6B^1D^2_f + 6B^1D^1_g + 3D^2_{ff} - 3D^2_{gg} - 6D^1_{fg} + 4C^2C^1_f - 4C^2C^2_g + \\ &\qquad 4C^1C^2_f - 4C^1C^1_g - 4C^2B^1_x - 4C^1B^2_x = 0. \end{aligned} \tag{39}$$

**Proof.** Suppose that there exists complex functions,

$$\begin{aligned}
A(x,u) &= A^1(x,f,g) + iA^2(x,f,g),\\
B(x,u) &= B^1(x,f,g) + iB^2(x,f,g),\\
C(x,u) &= C^1(x,f,g) + iC^2(x,f,g),\\
D(x,u) &= D^1(x,f,g) + iD^2(x,f,g),
\end{aligned} \tag{40}$$

such that the above system can be mapped into the second-order ODE

$$u''(x) = A(x,u)u'^3 + B(x,u)u'^2 + C(x,u)u' + D(x,u), \tag{41}$$

which is at most cubic in $u'$ and therefore satisfies the necessary condition of linearizability. To check the sufficient conditions the set of equations (39) is transformed into the equations

$$\begin{aligned}
&3A_{xx} + 3A_xC + 3AC_x - 3A_uD + C_{uu} - 6AD_u + BC_u - 2BB_x - 2B_{xu} = 0,\\
&6A_xD - 3B_uD + 3AD_x + B_{xx} - 2C_{xu} - 3BD_u + 3D_{uu} + 2CC_u - CB_x = 0,
\end{aligned} \tag{42}$$

which are the Lie conditions. Since the ODE (41) is linearizable, we can obtain its solution by the geometric method. This solution can now be written as the pair of real functions, $f$ and $g$. Hence the system of ODEs (38) can be solved by complex linearization.

To see how the complex variable approach works we present some illustrative examples of two dimensional systems of ODEs.

**Examples**

**1.** Consider the system of ODEs

$$\begin{aligned}
ff'' - gg'' &= f'^2 - g'^2 + (f^2 - g^2)w_1(x) - 2fgw_2(x),\\
fg'' + gf'' &= 2f'g' + 2fgw_1(x) + (f^2 - g^2)w_2(x),
\end{aligned} \tag{43}$$

where $w_1$ and $w_2$ are arbitrary functions of the time variable $x$. This was discussed in [12] for an anharmonic oscillator with $w_1 = x^2$ and $w_2 = 0$. The complex symmetry analysis gave interesting insights into the oscillator dynamics. Since the coefficients satisfy the linearizability conditions therefore the above system is solvable by complex linearization. In fact the system is linearizable. The transformation for our purpose is

$$\chi = \frac{x}{x^2 + f^2}, \quad \Upsilon = \frac{1}{2x}\ln(f^2 + g^2), \quad \zeta = \frac{1}{x}\arctan\left(\frac{g}{f}\right), \tag{44}$$

and it reduces system (43) into the linear system of ODEs

$$\Upsilon'' = \frac{1}{\chi}w_1, \quad \zeta'' = \frac{1}{\chi}w_2, \tag{45}$$

where

$$w_1 \equiv w_1(1/\chi), \quad w_2 \equiv w_2(1/\chi). \tag{46}$$

**2.** *Two-dimensional Coupled Modified Emden System:*

The modified Emden equation possess numerous dynamical properties in nonlinear oscillations. In [8] *Chandrasekar et al.* explore an important characteristic of such an equation in which the frequency of oscillation is independent of amplitude and remains the same as that of the linear oscillator. They showed that the amplitude dependence of the frequency is not a fundamental property of nonlinear dynamical phenomena. Later, in [15], they extended the above results and investigated the dynamical properties of N-coupled nonlinear oscillator

of Lienard type. Those systems which possess a Hamiltonian structure can be transformed into systems of uncoupled harmonic oscillators via contact transformations. We investigate the integrability of a system of two cubically semilinear coupled ODEs of Emden type

$$
\begin{aligned}
f'' &= -3ff' + 3gg' - f^3 + 3fg^2, \\
g'' &= -3gf' - 3fg' - 3f^2g + g^3.
\end{aligned} \tag{47}
$$

The coefficients satisfy the linearizability conditions. Hence the above system is solvable by complex linearization. This system corresponds to the cubically semilinear modified Emden ODE

$$u'' + 3uu' + u^3 = 0, \tag{48}$$

which is linearizable as it satisfies the Lie conditions. This equation arises in many applications (see e.g. [24, 25]). Equation (48) admits the two noncommuting complex symmetries

$$\mathbf{Z}_1 = \frac{\partial}{\partial x}, \quad \mathbf{Z}_2 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}, \tag{49}$$

which can be used to write down the transformation,

$$\chi = x - \frac{1}{u}, \quad U = \frac{x^2}{2} - \frac{x}{u}, \tag{50}$$

that transforms (48) into the free-particle equation $U'' = 0$. This transformation seems odd as $x$ is real while $\chi$ is complex. The point is that we start and end with a complex independent variable *restricted to the real line*, but in the intervening steps the variable moves off it. The procedure is reminiscent of analytic continuation. To check its consistency we express the solution of (47) in the new coordinates $(\chi, U)$

$$U = \alpha\chi + \beta, \tag{51}$$

where $\alpha$ and $\beta$ are complex constants. In coordinates $(x, u)$ the above equation yields

$$u = \frac{2(x-\alpha)}{x^2 - 2\alpha x - 2\beta}, \tag{52}$$

which satisfies (48). It generates the solution of the system (47)

$$
\begin{aligned}
f &= \frac{2(x-\alpha_1)(x^2 - 2\alpha_1 x - 2\beta_1) + 4\alpha_2(\alpha_2 x + \beta_2)}{(x^2 - 2\alpha_1 x - 2\beta_1)^2 + (2\alpha_2 x + 2\beta_2)^2}, \\
g &= \frac{4(x-\alpha_1)(\alpha_2 x + \beta_2) - 2\alpha_2(x^2 - 2\alpha_1 x - 2\beta_1)}{(x^2 - 2\alpha_1 x - 2\beta_1)^2 + (2\alpha_2 x + 2\beta_2)^2},
\end{aligned} \tag{53}
$$

which would have been difficult to obtain by other means. Note that here the C-REs are no longer preserved by the transformation.

**3.** Now consider the Newtonian system of ODEs with velocity dependent forces

$$
\begin{aligned}
f'' &= 1 + \left((f'-x)^2 - g'^2\right) w_1 - 2(f'-x)g'w_2, \\
g'' &= 2(f'-x)g'w_1 + \left((f'-x)^2 - g'^2\right) w_2,
\end{aligned} \tag{54}
$$

where

$$w_1 \equiv w_1(2f - x^2, 2g), \quad w_2 \equiv w_2(2f - x^2, 2g). \tag{55}$$

It can be verified that system (54) can be solved by complex linearization as it satisfies the criteria of Theorem 3. It corresponds to the complex Newtonian equation

$$u'' = 1 + (u' - x)^2 w(2u - x^2) \tag{56}$$

with quadratic velocity dependent forces. This ODE admits the complex Lie symmetries

$$\mathbf{Z}_1 = \frac{\partial}{\partial x} + x\frac{\partial}{\partial u}, \quad \mathbf{Z}_2 = x\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial u}. \tag{57}$$

Equation (56) is linearized by the complex transformation

$$\chi = 2u - x^2, \quad U = x, \tag{58}$$

to become

$$2U'' = -U'w(\chi). \tag{59}$$

Again (58) is reminiscent of analytic continuation as it is from (real, complex) to (complex, real). To check consistency we may take $w = 1$, i.e., $w_1 = 1, w_2 = 0$, to obtain

$$u = \alpha + \ln 2 + \frac{x^2}{2} - \ln(\beta - x), \tag{60}$$

where $\alpha$ and $\beta$ are complex constants. Putting $w_1 = 1$ and $w_2 = 0$ in (54) we get the system

$$\begin{aligned} f'' &= 1 + x^2 + f'^2 - g'^2 - 2xf', \\ g'' &= 2f'g' - xg', \end{aligned} \tag{61}$$

with the general solution

$$\begin{aligned} f &= \alpha_1 - \ln 2 + \frac{x^2}{2} - \frac{1}{2}\ln\left((\beta_1 - x)^2 + \beta_2^2\right), \\ g &= \alpha_2 - \arctan\left(\frac{\beta_2}{\beta_2 - x}\right). \end{aligned} \tag{62}$$

Also, if we take

$$w(2u - x^2) = \frac{1}{2u - x^2}, \tag{63}$$

the solution of (56) is

$$u = \frac{x^2}{2} + \sqrt{\frac{\beta - x}{2\alpha}}, \tag{64}$$

which gives the solution

$$\begin{aligned} f(x) &= \frac{x^2}{2} + R(x)\cos(\theta(x)), \\ g(x) &= R(x)\sin\left(\theta(x)\right), \end{aligned} \tag{65}$$

where

$$\begin{aligned} R(x) &= \sqrt{\frac{(\alpha_1(\beta_1 - x) + \beta_2\alpha_2)^2 + (\beta_2\alpha_2 - \alpha_2(\beta_1 - x))^2}{2(\alpha_1^2 + \alpha_2^2)}}, \\ \theta(x) &= \frac{\beta_2\alpha_2 - \alpha_2(\beta_1 - x)}{2(\alpha_1(\beta_1 - x) + \beta_2\alpha_2)}, \end{aligned}$$

of the system

$$f'' = 1 + \frac{(2f-x^2)((f'-x)^2-g'^2)}{(2f-x^2)^2+4g^2} + \frac{4gg'(f'-x)}{(2f-x^2)^2+4g^2},$$
$$g'' = \frac{2g'(f'-x)(2f-x^2)}{(2f-x^2)^2+4g^2} - \frac{2g((f'-x)^2-g'^2)}{(2f-x^2)^2+4g^2}. \tag{66}$$

Note that $w$ is an arbitrary complex function that gives a class of systems of ODEs that correspond to (56). Thus the linearization of a general equation encodes the linearization of a large class of systems of ODEs.

## 5. Conclusion and Discussion

Though the linearizaton procedure for a scalar ODE was fully provided by Lie, there is no such complete characterization and procedure available more generally [5, 6, 21], [24]−[26]). Geometry gives a procedure that not only provides the invariant characterization but also the solution of systems of ODEs [30, 31]. However, it only applies to the class of maximum symmetry and it is known that some less symmetric systems of ODEs are also linearizable [39]. A method that retains the power of geometry, but applies to the less symmetric cases is needed.

In this paper we used complex scalar ODEs to write equivalent systems of PDEs and ODEs and then required that the original ODEs be linearizable. We provided various examples, mainly of mechanical systems, to illustrate the power and use of the method presented. For the PDEs we have a guarantee that the C-REs are preserved under the linearizing transformation. However, for the ODEs we have no such guarantee. It seems that, when the C-REs are preserved, we obtain linearizing transformations and when they are not preserved we are not able to linearize the system. However, the solution of the system obtained by linearization of the complex scalar ODE is still valid for the equivalent system of PDEs or ODEs.

We called the procedure for ODEs solution by complex linearization. This is an example of Penrose's "complex magic" [34]. The solutions are again obtained using the geometric linearization procedure. It is hoped that the complex procedures, augmenting the real linearization for maximally symmetric systems, will give all the classes of linearizable systems of ODEs. This line of investigation is being pursued [3, 36].

For PDEs we get a systematic procedure to linearize the system and obtain the solution. However, it is clear that we do not get all possible solutions. This can be proved by considering the linear complex ODE, $u''=0$, and writing the corresponding system of PDEs,

$$f_{xx} - f_{yy} + 2g_{xy} = 0,$$
$$g_{xx} - g_{yy} - 2f_{xy} = 0. \tag{67}$$

Since there are only two arbitrary (complex) constants that can appear for the scalar ODE but infinitely many linearly independent solutions of the system of PDEs, we see that the complex linearization procedure cannot exhaust the solutions of the system of PDEs.

One should be able to extend the complex linearization procedure to third- and fourth-order systems by using the results for the corresponding scalar ODEs [19, 20] straightforwardly. Also the extension to conditional linearizability of systems could be obtained [27, 28, 29]. However, in the former case the power of the geometric approach would be lost.

In the latter case, if the root equation is second-order, the geometric method would provide the solution of the system.

A bigger problem is the extension to higher-dimensional systems. The extension to dimensions of $2n$ can be obtained by iterative use of the complex method. Starting with a real system of $n$-dimensions and converting to complex variables, we could get $2n$-dimensional systems as well. However, it would appear that the iterative procedure and the complexification of the system would yield different linearizable classes in general. Furthermore, this would not provide a means of dealing with systems of odd dimensions. It would be interesting to explore the various ramifications of extension of complex linearization to higher dimensions.

## Acknowledgments

SA is most grateful to NUST and DECMA in providing financial assistance for his stay at the University of the Witwatersrand, Johannesburg, South Africa, where this work was initiated. AQ acknowledges the School of Computational and Applied Mathematics (DECMA) for funding his stay at the university.

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